

ON COTYPE AND A GROTHENDIECK-TYPE THEOREM FOR ABSOLUTELY SUMMING MULTILINEAR OPERATORS

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ABSTRACT. A famous result due to Grothendieck asserts that every continuous linear operator from ℓ_1 to ℓ_2 is absolutely $(1, 1)$ -summing. If $n \geq 2$, however, it is very simple to prove that every continuous n -linear operator from $\ell_1 \times \cdots \times \ell_1$ to ℓ_2 is absolutely $(1; 1, \dots, 1)$ -summing, and even absolutely $(\frac{2}{n}; 1, \dots, 1)$ -summing. In this note we deal with the following problem:

Given a positive integer $n \geq 2$, what is the best constant $g_n > 0$ so that every n -linear operator from $\ell_1 \times \cdots \times \ell_1$ to ℓ_2 is absolutely $(g_n; 1, \dots, 1)$ -summing?

We prove that $g_n \leq \frac{2}{n+1}$ and also obtain an optimal improvement of previous recent results (due to Heinz Juenk *et al*, Geraldo Botelho *et al* and Dumitru Popa) on inclusion theorems for absolutely summing multilinear operators.

1. INTRODUCTION

Grothendieck's theorem for absolutely summing operators asserts that every continuous linear operator from ℓ_1 to ℓ_2 is absolutely $(1; 1)$ -summing (and hence absolutely $(p; p)$ -summing for every $p \geq 1$). For the linear theory of absolutely summing operators we refer to [13, 17] (see also [12, 19, 27] for recent developments).

In the multilinear setting, D. Pérez-García, in his PhD thesis [25] (see also [8] and [11] for a different proof), proved that every continuous n -linear operator from $\ell_1 \times \cdots \times \ell_1$ to ℓ_2 is multiple $(1; 1, \dots, 1)$ -summing (in fact, multiple $(p; p, \dots, p)$ -summing for every $1 \leq p \leq 2$). This result can be regarded as the multilinear version of Grothendieck's theorem.

Let us recall the notions.

The letters X_1, \dots, X_n, X, Y will always denote Banach spaces over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} and X^* represents the topological dual of X .

For any $s > 0$, we denote the conjugate of s by s^* . Given a positive integer n , the space of all continuous n -linear operators from $X_1 \times \cdots \times X_n$ to Y endowed with the sup norm is denoted by $\mathcal{L}(X_1, \dots, X_n; Y)$. For $p > 0$, the vector space of all sequences $(x_j)_{j=1}^\infty$ in X such that

$$\left\| (x_j)_{j=1}^\infty \right\|_p = \left(\sum_{j=1}^\infty \|x_j\|^p \right)^{\frac{1}{p}} < \infty$$

is denoted by $\ell_p(X)$. We represent by $\ell_p^w(X)$ the linear space of the sequences $(x_j)_{j=1}^\infty$ in X such that $(\varphi(x_j))_{j=1}^\infty \in \ell_p(\mathbb{K})$ for every $\varphi \in X^*$.

If $0 < p, q_1, \dots, q_n < \infty$ and $\frac{1}{p} \leq \frac{1}{q_1} + \cdots + \frac{1}{q_n}$, a multilinear operator $T \in \mathcal{L}(X_1, \dots, X_n; Y)$ is absolutely $(p; q_1, \dots, q_n)$ -summing if $(T(x_j^{(1)}, \dots, x_j^{(n)}))_{j=1}^\infty \in \ell_p(Y)$ for every $(x_j^{(k)})_{j=1}^\infty \in \ell_{q_k}(X_k)$.

$\ell_{q_k}^w(X_k)$, $k = 1, \dots, n$. In this case we write $T \in \Pi_{(p; q_1, \dots, q_n)}^n(X_1, \dots, X_n; Y)$. For details we refer to [1].

When $1 \leq q_1, \dots, q_n \leq p < \infty$ a multilinear operator $T \in \mathcal{L}(X_1, \dots, X_n; Y)$ is multiple $(p; q_1, \dots, q_n)$ -summing if $(T(x_{j_1}^{(1)}, \dots, x_{j_n}^{(n)}))_{j_1, \dots, j_n=1}^\infty \in \ell_p(Y)$ for every $(x_j^{(k)})_{j=1}^\infty \in \ell_{q_k}^w(X_k)$, $k = 1, \dots, n$. In this case we write $T \in \Pi_{(p; q_1, \dots, q_n)}^n(X_1, \dots, X_n; Y)$. For details we mention [8, 21] and for recent developments and applications related to the multilinear and polynomial theory we refer to [2, 3, 7, 14, 15, 16, 20, 22, 24] and references therein. For $n = 1$ we write Π instead of Π^1 and we recover the classical theory of absolutely summing linear operators.

For $1 \leq q_1, \dots, q_n \leq p < \infty$, the inclusion

$$\Pi_{(p; q_1, \dots, q_n)}^n(X_1, \dots, X_n; Y) \subseteq \Pi_{(p; q_1, \dots, q_n)}^n(X_1, \dots, X_n; Y)$$

is obvious. So, the following coincidence result is an immediate consequence of Pérez-García multilinear version of Grothendieck's theorem:

Theorem 1.1. *For every positive integer n ,*

$$\Pi_{(1; 1, \dots, 1)}^n(\ell_1, \dots, \ell_1; \ell_2) = \mathcal{L}(\ell_1, \dots, \ell_1; \ell_2).$$

However, using that ℓ_1 has cotype 2 it is easy to prove that the above result is far from being optimal. In fact, we have the following improvement (see [9, 23]):

Theorem 1.2. *For every positive integer $n \geq 2$,*

$$(1.1) \quad \Pi_{(\frac{2}{n}; 1, \dots, 1)}^n(\ell_1, \dots, \ell_1; \ell_2) = \mathcal{L}(\ell_1, \dots, \ell_1; \ell_2).$$

So, the following problem is quite natural:

Problem 1.3. *Given a positive integer $n \geq 2$, what is the best constant $g_n > 0$ so that*

$$\Pi_{(g_n; 1, \dots, 1)}^n(\ell_1, \dots, \ell_1; \ell_2) = \mathcal{L}(\ell_1, \dots, \ell_1; \ell_2)?$$

If we test $n = 1$ in (1.1) we obtain

$$\Pi_{(2; 1)}(\ell_1; \ell_2) = \mathcal{L}(\ell_1; \ell_2)$$

which is not surprising at all, in view of Grothendieck's Theorem. So, in some sense, we feel that the estimate $g_n \leq \frac{2}{n}$ for $n \geq 2$ is probably not optimal. The optimistic reader will probably hope for an estimate for g_n so that in the case $n = 1$ we recover Grothendieck's Theorem. Fortunately, in the last section we will precisely obtain such an estimate.

The problem of estimating g_n is related to the generalization of certain results involving cotype and absolutely summing multilinear operators. The following result is a combination of [18, Theorem 3 and Remark 2], [26, Corollary 4.6] and [10, Theorem 3.8 (ii)]:

Theorem 1.4 (Inclusion Theorem). *Let X_1, \dots, X_n be Banach spaces with cotype s and $n \geq 2$ be a positive integer:*

(i) *If $s = 2$, then*

$$(1.2) \quad \Pi_{(q; q, \dots, q)}^n(X_1, \dots, X_n; Y) \subseteq \Pi_{(p; p, \dots, p)}^n(X_1, \dots, X_n; Y)$$

holds true for $1 \leq p \leq q \leq 2$ and every Y .

(ii) If $s > 2$, then

$$(1.3) \quad \Pi_{(q;q,\dots,q)}^n(X_1, \dots, X_n; Y) \subseteq \Pi_{(p;p,\dots,p)}^n(X_1, \dots, X_n; Y)$$

holds true for $1 \leq p \leq q < s^*$ and every Y .

The results above are clearly not always optimal since, for example,

$$\Pi_{(2;2,2,2)}^3(\ell_2, \ell_2, \ell_2; \mathbb{K}) \neq \mathcal{L}(\ell_2, \ell_2, \ell_2; \mathbb{K}) = \Pi_{(\frac{2}{3};1,1,1)}^3(\ell_2, \ell_2, \ell_2; \mathbb{K}).$$

So, another natural problem is:

Problem 1.5. Given $1 \leq p \leq q < \infty$ and a positive integer $n \geq 2$, what are the optimal $\alpha := \alpha_{p,q,n} > 0$ so that, under the same circumstances of (1.2) and (1.3), we have

$$(1.4) \quad \Pi_{(q;q,\dots,q)}^n(X_1, \dots, X_n; Y) \subseteq \Pi_{(\alpha;p,\dots,p)}^n(X_1, \dots, X_n; Y)$$

for all Banach spaces X_1, \dots, X_n, Y ?

In this direction we extend Theorem 1.4 and also recent results from [4, 5] by showing that

$$\alpha \leq \frac{qp}{n(q-p) + p}$$

and, in some sense, this constant is optimal, since for this value of α we have an equality in (1.4).

2. AN ESTIMATE FOR α

Theorem 2.1. Let $1 \leq k \leq n$, where $n \geq 2$ is a positive integer. If X_i has cotype $s_i \geq 2, i = 1, \dots, k$ and

$$1 \leq p \leq q < \min_{1 \leq i \leq k} s_i^* \text{ if } s_i > 2 \text{ for some } i = 1, \dots, k$$

or

$$1 \leq p \leq q \leq 2 \text{ if } s_i = 2 \text{ for all } i = 1, \dots, k,$$

then

$$\Pi_{(z;q,\dots,q,t,\dots,t)}^n(X_1, \dots, X_n; Y) = \Pi_{(\frac{zqp}{zk(q-p)+qp};p,\dots,p,t,\dots,t)}^n(X_1, \dots, X_n; Y),$$

for all X_{k+1}, \dots, X_n, Y and all $z, t \geq 1$ (here q and p are repeated k times). In particular, if $k = n$,

$$\Pi_{(z;q,\dots,q)}^n(X_1, \dots, X_n; Y) = \Pi_{(\frac{zqp}{zk(q-p)+qp};p,\dots,p)}^n(X_1, \dots, X_n; Y)$$

Proof. Since X_i has finite cotype $s_i \geq 2, i = 1, \dots, k$, then we have

$$\ell_p^w(X_i) = \ell_{qp/(q-p)}^w \ell_q^w(X_i)$$

for all $i = 1, \dots, k$ with

$$1 \leq p \leq q < \min_{1 \leq i \leq k} s_i^* \text{ if } s_i > 2 \text{ for some } i = 1, \dots, k$$

or

$$1 \leq p \leq q \leq 2 \text{ if } s_i = 2 \text{ for all } i = 1, \dots, k.$$

Let $(x_j^{(i)})_{j=1}^\infty \in \ell_p^w(X_i), i = 1, \dots, k$ and $(x_j^{(i)})_{j=1}^\infty \in \ell_t^w(X_i)$ for $i = k + 1, \dots, n$. So $x_j^{(i)} = \alpha_j^{(i)} y_j^{(i)}$, with $(\alpha_j^{(i)})_{j=1}^\infty \in \ell_{qp/(q-p)}$ and $(y_j^{(i)})_{j=1}^\infty \in \ell_q^w(X_i)$, for all j and $i = 1, \dots, k$. If $A \in \Pi_{(z; q, \dots, q, t, \dots, t)}^n(X_1, \dots, X_n; Y)$, then

$$\begin{aligned} & \left(\sum_{j=1}^\infty \left\| A \left(x_j^{(1)}, \dots, x_j^{(n)} \right) \right\|^{\frac{zqp}{zk(q-p)+qp}} \right)^{\frac{zk(q-p)+qp}{zqp}} \\ &= \left(\sum_{j=1}^\infty \left(\left| \alpha_j^{(1)} \cdots \alpha_j^{(k)} \right| \left\| A \left(y_j^{(1)}, \dots, y_j^{(k)}, x_j^{(k+1)}, \dots, x_j^{(n)} \right) \right\|^{\frac{zqp}{zk(q-p)+qp}} \right)^{\frac{zk(q-p)+qp}{zqp}} \right. \\ &\leq \left(\sum_{j=1}^\infty \left\| A \left(y_j^{(1)}, \dots, y_j^{(k)}, x_j^{(k+1)}, \dots, x_j^{(n)} \right) \right\|^z \right)^{\frac{1}{z}} \left(\sum_{j=1}^\infty \left| \alpha_j^{(1)} \cdots \alpha_j^{(k)} \right|^{\frac{qp}{k(q-p)}} \right)^{k \left(\frac{q-p}{qp} \right)} \\ &\leq \left(\sum_{j=1}^\infty \left\| A \left(y_j^{(1)}, \dots, y_j^{(k)}, x_j^{(k+1)}, \dots, x_j^{(n)} \right) \right\|^z \right)^{\frac{1}{z}} \prod_{i=1}^k \left(\sum_{j=1}^\infty \left| \alpha_j^{(i)} \right|^{\frac{qp}{(q-p)}} \right)^{\frac{q-p}{qp}} < \infty \end{aligned}$$

and we conclude that

$$\Pi_{(z; q, \dots, q, t, \dots, t)}^n(X_1, \dots, X_n; Y) \subseteq \Pi_{(\frac{zqp}{zk(q-p)+qp}; p, \dots, p, t, \dots, t)}^n(X_1, \dots, X_n; Y).$$

The other inclusion is a consequence of the inclusion theorem for absolutely summing multilinear operators. \square

A similar result holds if $X_{j_1}, \dots, X_{j_k}, \{j_1, \dots, j_k\} \subseteq \{1, \dots, n\}$ (instead of X_1, \dots, X_k) have cotype $s_{j_i} \geq 2, i = 1, \dots, k$.

The following immediate corollary is an optimal (in the sense that we have an equality instead of an inclusion) generalization of Theorem 1.4:

Corollary 2.2. *If $n \geq 2$ and X_1, \dots, X_n have finite cotype s and*

$$1 \leq p \leq q < s^* \text{ if } s > 2$$

or

$$1 \leq p \leq q \leq 2 \text{ if } s = 2,$$

then

$$\Pi_{(q; q, \dots, q)}^n(X_1, \dots, X_n; Y) = \Pi_{(\frac{qp}{n(q-p)+p}; p, \dots, p)}^n(X_1, \dots, X_n; Y)$$

for every Banach space Y and

$$\alpha \leq \frac{qp}{n(q-p)+p}.$$

Remark 2.3. *The above results were independently proved in [6].*

3. AN ESTIMATE FOR g_n

From Corollary 2.2 we know that

$$\Pi_{(2;2,\dots,2)}^n(\ell_1, \dots, \ell_1; \ell_2) = \Pi_{(\frac{2}{n+1};1,\dots,1)}^n(\ell_1, \dots, \ell_1; \ell_2)$$

for all $n \geq 2$. But, since

$$\mathcal{L}(\ell_1, \dots, \ell_1; \ell_2) = \Pi_{m(2;2,\dots,2)}^n(\ell_1, \dots, \ell_1; \ell_2) \subseteq \Pi_{(2;2,\dots,2)}^n(\ell_1, \dots, \ell_1; \ell_2)$$

it readily follows that

$$\Pi_{(\frac{2}{n+1};1,\dots,1)}^n(\ell_1, \dots, \ell_1; \ell_2) = \mathcal{L}(\ell_1, \dots, \ell_1; \ell_2)$$

for all $n \geq 2$. So we have:

Theorem 3.1. *If $n \geq 2$, then*

$$g_n \leq \frac{2}{n+1}.$$

Note that Grothendieck's Theorem asserts that $g_1 = 1$ and $1 = \frac{2}{1+1}$; hence we conjecture that $\frac{2}{n+1}$ is in fact the optimal estimate for g_n .

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